

# Remarks on a theorem of Perron <sup>1</sup>

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## Abstract

For a linear differential equation with a mild condition on its singularities, we discuss generalized continued fractions converging to expressions in its solutions and their derivatives. In the case of an order two linear differential equation, this is the logarithmic derivative of the holomorphic solution near a singularity.

## 1 Introduction

In [1] Ince analyzed the process introduced by Perron [2] leading to find solutions of the Gauss hypergeometric equation:

$$(1) \quad z(1-z)y'' + (c - (a+b+1)z)y' - aby = 0, \quad a, b, c \in \mathbb{C} - \mathbb{Z}.$$

by means of successive differentiation of (1). This process is equivalent to the set of recurrence relations

$$(2) \quad x_n = a_{1,n}x_{n+1} + a_{2,n}x_{n+2}, \text{ where } x_n = \frac{y^{(n)}}{n!}, \quad n = 0, 1, 2, \dots$$

and

$$a_{1,n} := \frac{(n+1)(c+n-(a+b+2n+1)z)}{(a+n)(b+n)},$$

$$a_{2,n} := \frac{(n+1)(n+2)z(z-1)}{(a+n)(b+n)}, \quad n = 1, 2, \dots$$

The corresponding continued fraction

$$(3) \quad \cfrac{1}{a_{1,0} + \cfrac{a_{2,0}}{a_{1,1} + \cfrac{a_{2,1}}{\ddots \cfrac{a_{2,n-3}}{a_{1,n-3} + \cfrac{a_{2,n-3}}{a_{1,n-2}}}}}$$

converges on one hand to  $(\ln F(a, b, c|z))'$  in the region  $\operatorname{Re}(z) < \frac{1}{2}$  and on the other hand to  $(\ln F(a, b, a+b-c+1|1-z))'$  in the region  $\operatorname{Re}(z) > \frac{1}{2}$ . In this paper we present a generalization of this process to second order Fuchsian differential equations using Poincaré's theorem on recurrence sequences. We would like to thank J. Oesterlé for many useful conversations and to Frits Beukers who indicated to us the relevance of the Poincaré's recurrence theorem for this matter.

<sup>1</sup> Keywords: Fuchsian differential equations, exponents of a singularity

## 2 Poincaré's theorem on recurrent sequences

In [4] Section 2 pages 213-217 and Section 6 page 237, Poincaré proved the following theorem.

**Theorem 1.** *For a fixed  $k \in \mathbb{N}$ , let us be given a recurrent sequence  $f_n$*

$$(4) \quad f_{n+k} + a_{k-1,n}f_{n+k-1} + \cdots + a_{1,n}f_{n+1} + a_{0,n}f_n = 0, \quad n \in \mathbb{N}_0,$$

where  $\lim_{n \rightarrow \infty} a_{j,n} = a_j < \infty$ . Assume that the roots of the characteristic equation  $z^k + \sum_{j=0}^{k-1} a_j z^j$  have different absolute values. Then either  $f_n = 0$  for large  $n$  or  $\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n}$  converges to a root of the characteristic equation.

Let  $f = \sum_{n=0}^{\infty} f_n z^n$  be a holomorphic solution of the linear differential equation  $L(f) = 0$ , where

$$(5) \quad L := P_0(\delta) + zP_1(\delta) + \cdots + z^k P_k(\delta)$$

$$(6) \quad = Q_0(z)\delta^m + Q_1(z)\delta^{m-1} + \cdots + Q_m(z)$$

where  $\delta = z \frac{d}{dz}$ . Comparing the coefficients of  $z^{n+k}$  in both sides of  $L(f) = 0$  we get the recurrence relation (4) with

$$a_{j,n} = \frac{P_{k-j}(n+j)}{P_0(n+k)}, \quad j = 0, 1, \dots, k-1.$$

(for  $z^j$ ,  $0 \leq j < k$ , we get restrictions on the coefficients  $f_0, f_1, \dots, f_{k-1}$ ). Any linear differential equation with a regular singularity at  $z = 0$  can be written in the format (5) with  $\deg(P_j) \leq \deg(P_0)$ ,  $j = 1, 2, \dots, k$ , and so for these differential equations  $\lim_{n \rightarrow \infty} a_{j,n}$  exists. It turns out that  $Q_0$  is the characteristic polynomial of the recurrence relation and  $Q_0(0) \neq 0$ . Note that  $P_0$  is the indicial equation of  $L$  at  $z = 0$  and for all except a finite number of points we have  $P_0(\delta) = \delta(\delta - 1) \cdots (\delta - (n - 1))$ .

**Definition 1.** For a linear differential equation  $L$ , we denote by  $A_L$  the set of real lines in  $\mathbb{C}$  perpendicular in the middle to the segments connecting the singularities of  $L$ .

Now, we are able to reformulate Poincaré's theorem in the following format.

**Theorem 2.** *Let  $L$  be a linear differential equation with regular singularities  $\{t_1, t_2, \dots, t_r\}$  at  $\mathbb{C}$ . For any holomorphic non-polynomial solution  $f = \sum_{n=0}^{\infty} f_n(z - z_0)^n$  around a point  $z_0 \in \mathbb{C} \setminus A_L$ , the limit  $\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n}$  exists and it is one of  $t_j$ 's.*

Note that in the above theorem  $\infty$  may be an irregular singularity. The mentioned  $t_j$  is a singularity such that  $f$  converges in a disc with center  $z_0$  and radius  $|t_j - z_0|$  and not beyond this disc. It may contain some singularities of  $L$ . Finally note also that if  $\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = t_j$  exists then  $\lim_{n \rightarrow \infty} f_n^{\frac{1}{n}} = t_j$ . For further references on Poincaré's theorem see [6]

## 3 A convergence theorem

Let us be given a linear differential equation

$$(7) \quad L: y^{(m)} = \sum_{i=0}^{m-1} q_i y^{(i)}, \quad q_0, q_1, \dots, q_{m-1} \in \mathbb{C}(z),$$

where  $\mathbb{C}(z)$  is the field of rational functions in  $z$  with complex coefficients, and let  $S = \{t_1, t_2, \dots, t_r\} \subset \mathbb{C}$  be the set of its singular points. We associate to  $L$  the open sets  $U_1, U_2, \dots, U_r$  defined by

$$(8) \quad U_i := \{z \in \mathbb{C} \mid |z - t_i| < |z - t_j|, j = 1, 2, \dots, r, j \neq i\}$$

The set  $\mathbb{C} \setminus \bigcup_{i=1}^r U_i$  consists of segment of lines perpendicular in the middle to the segments connecting two points  $t_{j_1}$  and  $t_{j_2}$ ,  $j_1, j_2 = 1, 2, \dots, r$ ,  $j_1 \neq j_2$ .

Note that  $t_i$  is the unique element of  $S$  inside  $U_i$ .

**Definition 2.** We say that  $L$  is generic if for each singularity  $t_i$  of (7) in the affine plane  $\mathbb{C}$  there is a basis  $y_1, y_2, \dots, y_m$  of the  $\mathbb{C}$ -vector space of its solutions such that near  $t_i$  the solutions  $y_2, \dots, y_m$  extend holomorphically to  $t_i$  and the solution  $y_1$  does not extends.

The reader may have noticed that we do not put any condition on the singularity  $\infty$ .

**Problem 1.** Let  $f_1, f_2$  be two solutions of a generic Fuchsian differential equation  $L$  in  $U_i$ . Then the limit

$$(9) \quad \lim_{n \rightarrow \infty} \frac{f_1^{(n)}(z)}{f_2^{(n)}(z)}, \quad z \in U_i$$

exists and it is a constant number depending on  $i$ .

The following Proposition gives a partial answer to this problem.

**Proposition 1.** Let  $f_1, f_2$  be two solutions of a generic Fuchsian differential equation  $L$ . Then in each connected component  $U$  of  $\mathbb{C} \setminus A_L$ , the limit

$$(10) \quad \lim_{n \rightarrow \infty} \frac{f_1^{(n)}(z)}{f_2^{(n)}(z)},$$

exists and it is a constant number depending on  $U$ .

*Proof.* Take  $y_1, y_2, \dots, y_m$  a basis of solutions of (7) associated to  $t_i$  as it is described in Definition 2. For  $z \in \mathbb{C} \setminus S$ , let  $\rho_j(z)$  be the maximum real number such that  $y_j$  is holomorphic in a disc with center  $z$  and radius  $\rho_j(z)$ . Indeed  $\rho_j(z)$  is the convergence radius of the Taylor series of  $y_j$  at  $z$ . By Cauchy-Hadamard theorem and Poincaré's theorem on recurrence relations, for  $z \in \mathbb{C} \setminus A_L$  we have

$$\rho_j(z) = \left( \lim_n \left( \frac{|y_j^{(n)}(z)|}{n!} \right)^{\frac{1}{n}} \right)^{-1}$$

( $\overline{\lim}$  is substituted by  $\lim$  which is a stronger statement.) Consider  $U_i$ ,  $i = 1, 2, \dots, r$ . Since  $y_1$  (resp.  $y_j$ ,  $j \neq 1$ ) is not (resp. is) holomorphic at  $t_i$  and  $|z - t_j| > |z - t_i|$ , we have  $\rho_1(z) = |z - t_i|$  and  $\rho_j(z) > |z - t_i|$ . It follows that

$$\lim_n \left( \frac{|y_j^{(n)}(z)|}{|y_1^{(n)}(z)|} \right)^{\frac{1}{n}} = \frac{\rho_1(z)}{\rho_j(z)} < 1$$

which implies that

$$(11) \quad \lim_{n \rightarrow \infty} \left( \frac{y_j^{(n)}(z)}{y_1^{(n)}(z)} \right) = 0.$$

Since  $f_1$  and  $f_2$  are linear combinations of  $y_j$ 's the result follows.  $\square$

We calculate the  $n$ -th derivative of  $y$

$$y^{(n)} = \sum_{i=0}^{m-1} q_{i,n} y^{(i)},$$

$$q_{i,n} \in \mathbb{C}(z), \quad i = 0, 1, \dots, m-1, n = m, m+1, \dots$$

**Theorem 3.** *For a generic Fuchsian differential equation (7), the fraction  $\frac{q_{k,n}}{q_{j,n}}$ ,  $k, j = 0, 1, \dots, m-1$ ,  $k \neq j$ , converges uniformly in compact subsets of each component of  $\mathbb{C} \setminus A_L$  as  $n$  goes to infinity.*

If Problem 1 is solved then the convergence in the above theorem will occur in each  $U_i$ .

*Proof.* Let  $y_1, y_2, \dots, y_m$  be a basis of the  $\mathbb{C}$ -vector space of solutions of (7) and let

$$Y^{(i)} = [y_1^{(i)}, y_2^{(i)}, \dots, y_m^{(i)}]^{\text{tr}}, \quad i = 1, 2, 3, \dots$$

$$Q_n = [q_{0,n}, q_{1,n}, \dots, q_{m-1,n}]^{\text{tr}}$$

We have

$$Y^{(n)} = [Y, Y', \dots, Y^{(m-1)}] Q_n,$$

Therefore

$$(12) \quad Q_n = [Y, Y', \dots, Y^{(m-1)}]^{-1} Y^{(n)}$$

Now, the theorem follows from Proposition 1. □

**Remark 1.** The matrix  $X := [Y, Y', \dots, Y^{(m-1)}]^{\text{tr}}$  is a fundamental system of the linear differential equation  $X' = AX$ , where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ q_0 & q_1 & q_2 & \cdots & q_{m-1} \end{pmatrix}$$

and so  $\tilde{X} = [Y, Y', \dots, Y^{(m-1)}]^{-1}$  is a fundamental system of  $\tilde{X}' = (-A^{\text{tr}})\tilde{X}$ .

## 4 Second order linear differential equations

Consider a linear differential equation of order two

$$(13) \quad y'' = q_0(z)y + q_1(z)y', \quad q_0, q_1 \in \mathbb{C}(z)$$

We calculate the  $n$ -th derivative of  $y$

$$y^{(n)} = q_{0,n}(z)y + q_{1,n}(z)y', \quad q_{i,n} \in \mathbb{C}(z), \quad i = 0, 1.$$

Theorem 3 in this case is:

**Theorem 4.** For a generic Fuchsian differential equation (7), the fraction  $\frac{q_{0,n}}{q_{1,n}}$  converges uniformly in compact subsets of  $U_i$  to  $-\frac{y'_2}{y_2}$ , where  $y_2$  is the unique (up to multiplication by a constant) holomorphic solution of (7) in  $U_i$ .

It is not hard to verify that  $-\frac{q_{0,n}}{q_{1,n}}$  is equal to the continued fraction (3), where  $a_{i,n}$ ,  $i = 0, 1$  are given by (4), see §5.

*Proof of Theorem 4.* The convergence follows from Theorem 3. For the case  $m = 2$  we continue the proof of Theorem 3. We have

$$[Y, Y']^{-1} = \frac{1}{y_1 y'_2 - y_2 y'_1} \begin{pmatrix} y'_2 & -y'_1 \\ -y_2 & y_1 \end{pmatrix}$$

and so  $\frac{q_{0,n}}{q_{1,n}}$  converges to  $-\frac{y'_2}{y_2}$ . In the next section we will see that  $\frac{q_{0,n}}{q_{1,n}}$  can be written as the continued fraction (3).  $\square$

## 5 Continued fractions

Let us consider the differential equation (7) and

$$(14) \quad x_n = a_{1,n}x_{n+1} + a_{2,n}x_{n+2} + \cdots + a_{m,n}x_{n+m}, \text{ where } x_n = \frac{y^{(n)}}{n!}, \quad n = 0, 1, 2, \dots$$

The recursive relations (14) gives us a monster continued fraction as follows: We write (14) in the following form:

$$\frac{x_{n+1}}{x_n} = \frac{1}{a_{1,n} + a_{2,n} \frac{x_{n+2}}{x_{n+1}} + a_{3,n} \frac{x_{n+3}}{x_{n+2}} \frac{x_{n+2}}{x_{n+1}} + \cdots + a_{m,n} \prod_{i=0}^{m-2} \frac{x_{n+m-i}}{x_{n+m-i-1}}}$$

Now we take  $\frac{x_1}{x_0}$  and write it in terms of  $\frac{x_i}{x_{i-1}}$ ,  $i = 2, \dots, m$ . In the next step we replace  $\frac{x_2}{x_1}$ , wherever it appears, with the term given by the above formula for  $n = 1$ . We repeat this procedure until infinity. The reader may have noticed that it is not possible to write down all these substitutions in a A4 paper for arbitrary  $m$ . For  $m = 2$  we get the usual continued fraction and only in this case we are able to analyze its convergence

Let us recall some notation from [2] §57 concerning continued fractions. Let  $a_n, b_{n-1}$ ,  $n = 1, 2, \dots$  be two sequences of complex numbers. We write

$$(15) \quad \frac{A_n}{B_n} = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots + \frac{a_n}{b_{n-1} + \frac{a_n}{b_n}}}}}$$

and we have

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \quad B_n = b_n B_{n-1} + a_n B_{n-2} \\ A_{-1} = 1, \quad B_{-1} = 0, \quad A_0 = b_0, \quad B_0 = 1,$$

If there is a sequence of numbers  $x_n$ ,  $n = 0, 1, 2, \dots$ , satisfying

$$x_n = b_n x_{n+1} + a_{n+1} x_{n+2}, \quad n = 0, 1, \dots$$

then for  $n \geq 1$  we have

$$(16) \quad \begin{aligned} x_0 &= A_{n-1}x_n + a_n A_{n-2}x_{n+1}, \\ x_1 &= B_{n-1}x_n + a_n B_{n-2}x_{n+1}, \end{aligned}$$

Now let us come back to our notations of linear differential equations of order two. The equalities (16) in our case implies that  $-\frac{q_{0,n}}{q_{1,n}}$  is the continued fraction (3).

**Remark 2.** After we wrote this note we became aware of a paper by Norlund [7] which in page 447 essentially asserts a solution to Problem 1 in full generality in the case of second order Fuchsian differential equations. However his proof has nontrivial gaps. Let us explain one of the cases which he considers. Assuming that the indicial equation of the second order linear differential equation  $L$  at  $z = t_i$  has two distinct roots  $\alpha_j$ ,  $j = 1, 2$ , in a neighborhood of  $t_i$  we have two solutions  $f_j$  which are asymptotic to  $y_j = (z - t_i)^{\alpha_j}$ . He uses

$$(17) \quad \lim_{n \rightarrow \infty} \frac{f_j^{(n)}(z)}{y_j^{(n)}(z)} = 1, \quad z \in U_i$$

and then he concludes that the limit (9) is equal to  $\lim_{n \rightarrow \infty} \frac{y_1^{(n)}(z)}{y_2^{(n)}(z)}$ . In general, his argument is that if  $f_j$  near  $t_i$  is asymptotic to  $y_j$  then (17) must hold. Note that for a Fuchsian differential equations we can take  $y_j$  as polynomials in  $\ln(z - t_i)$  and  $(z - t_i)^{\alpha_i}$  and so calculating  $y_j^{(n)}$  is easy and explicit. It is not clear however why this kind of assertion must be true or, in other words, which kind of generic conditions we have to put on  $L$  such that (17) holds. For the asymptotic behavior of  $f_j^{(n)}$  for growing  $n$ , Norlund refers to the work of Perron [3].

## References

- [1] E. L. Ince, On the continued fractions connected with the hypergeometric equation. Proc. Lond. Math. Soc. (2) 18, 236-248, 1919.
- [2] O. Perron, Die Lehre von den Kettenbrüchen. Band II. Analytisch- funktionentheoretische Kettenbrüche. Stuttgart: B. G. Teubner Verlagsgesellschaft VI, 316 S. (1957).
- [3] O. Perron, Über das verhalten von  $f^{(v)}(x)$  für  $\lim v = \infty$  wenn  $f(x)$  einer linearen homogenen Differentialgleichung genügt. Stzgsber. Akad. München (math. Phys.), 1913, p. 355-382.
- [4] H. Poincaré, Sur les equations linéaires aux différentielles ordinaires et aux différences finies. American Journal of Mathematics, Vol. 7, No. 3, Apr., 1885.
- [5] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. *From Gauss to Painlevé. Aspects of Mathematics*, E16. Friedr. Vieweg & Sohn, Braunschweig, 1991. A modern theory of special functions.
- [6] A. Máté, Paul Nevai, A generalization of Poincaré's theorem for recurrence equations Journal of Approximation Theory, Volume 63, Issue 1, 1990.
- [7] N. E. Norlund Vorlesungen über Differenzenrechnung. Springer Verlag, 1924.